# 財務數學 11.7.1

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# 11.7 Pricing a European Call in a Jump Model

Pricing a European call when the underlying asset is a jump process.

Two cases：

11.7.1 The underlying asset is driven by a single Poisson process

11.7.2 The underlying asset is driven by a Brownian motion and a compound Poisson process.

The market is complete in the first case and incomplete in the second.

# 11.7.1 Asset Driven by a Poisson Process

Consider a stock modeled as a geometric Poisson process

$$S(t) = S(0)\exp\{\alpha t + N(t)\log(\sigma+1) - \lambda\sigma t\} = S(0)e^{(\alpha-\lambda\sigma)t}(\sigma+1)^{N(t)}$$

where $\sigma > -1$, $\sigma \neq 0$, and $N(t)$ is a Poisson process with intensity $\lambda > 0$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$

for which the differential is

$$dS(t) = \alpha S(t)\,dt + \sigma S(t-)\,dM(t).$$

$M(t) = N(t) - \lambda t$ is the compensated Poisson process.

## Change to a risk-neutral measure $\widetilde{\mathbb{P}}$

$$\widetilde{\mathbb{P}}(A) = \int_A Z(T)\, d\mathbb{P} \text{ for all } A \in \mathcal{F}, \text{ where } Z(t) = e^{(\lambda-\tilde{\lambda})t}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(t)}.$$

$$dS(t) = rS(t)\,dt + \sigma S(t-)\,d\widetilde{M}(t), \quad \widetilde{M}(t) = N(t) - \tilde{\lambda}t \quad \tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}.$$

$$S(t) = S(0)e^{(r-\tilde{\lambda}\sigma)t}(\sigma+1)^{N(t)}.$$

**In order to rule out arbitrage, we must assume $\tilde{\lambda} > 0$, which is equivalent to**

$$\lambda > \frac{\alpha - r}{\sigma}.$$

$$dS(t) = rS(t)\,dt + \sigma S(t-)\,d\widetilde{M}(t)$$

**is equivalent to** $d\left(e^{-rt}S(t)\right) = \sigma e^{-rt}S(t-)\,d\widetilde{M}(t).$

$Y(t, S(t)) \equiv e^{-rt}S(t)$

by Ito's lemma

$$d(e^{-rt}S(t)) = dY(t, S(t)) = \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial S}dS + \frac{1}{2}\frac{\partial^2 Y}{\partial S^2}(dS)^2$$

$$= -re^{-rt}S(t)\,dt + e^{-rt}\left(rS(t)\,dt + \sigma S(t-)\,d\widetilde{M}(t)\right) + 0$$

$$= \sigma e^{-rt}S(t-)\,d\widetilde{M}(t)$$

**Since $\widetilde{M}(t)$ is a martingale, the discounted asset price is a martingale under $\widetilde{\mathbb{P}}$.**

# Pricing Formula for risk-neutral price of a European Call

**For $0 \le t \le T$, let $V(t)$ denote the risk-neutral price of a European call paying $V(t) = (S(T) - K)^+$ at time T.**

**The discounted call price**

$$e^{-rt}V(t) = \widetilde{\mathbb{E}}\left[e^{-rT}V(T)\middle|\mathcal{F}(t)\right] = \widetilde{\mathbb{E}}\left[e^{-rT}\left(S(T) - K\right)^+\middle|\mathcal{F}(t)\right].$$

**is a martingale under the risk-neutral measure.**

$$\Rightarrow V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(T) - K\right)^+\middle|\mathcal{F}(t)\right]$$

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(T)-K\right)^{+} \middle| \mathcal{F}(t)\right]$$

$$\begin{aligned} S(T) &= S(0)e^{(r-\tilde{\lambda}\sigma)T}(\sigma+1)^{N(T)} \\ &= S(0)e^{(r-\tilde{\lambda}\sigma)t}(\sigma+1)^{N(t)} \cdot e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)} \\ &= S(t) \cdot e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)}. \end{aligned}$$

$$\Rightarrow \; V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(t)e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)} - K\right)^{+} \middle| \mathcal{F}(t)\right]$$

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(t)e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)} - K\right)^{+}\Big|\mathcal{F}(t)\right]$$

**Lemma 2.3.4 (Independence).** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Suppose the random variables $X_1, \ldots, X_K$ are $\mathcal{G}$-measurable and the random variables $Y_1, \ldots, Y_L$ are independent of $\mathcal{G}$. Let $f(x_1, \ldots, x_K, y_1, \ldots, y_L)$ be a function of the dummy variables $x_1, \ldots, x_K$ and $y_1, \ldots, y_L$, and define*

$$g(x_1, \ldots, x_K) = \mathbb{E}f(x_1, \ldots, x_K, Y_1, \ldots, Y_L). \tag{2.3.27}$$

*Then*

$$\mathbb{E}[f(X_1, \ldots, X_K, Y_1, \ldots, Y_L)|\mathcal{G}] = g(X_1, \ldots, X_K). \tag{2.3.28}$$

The random variable $S(t)$ is $\mathcal{F}(t)$-measurable, whereas

$e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)}$ is independent of $\mathcal{F}(t)$.

**Replacing S(t) by dummy variable x.** $c(t,x) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(xe^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)} - K\right)^{+}\right]$

**then,** $V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(t)e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)} - K\right)^{+}\Big|\mathcal{F}(t)\right] = c(t, S(t))$

$$c(t,x) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(xe^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)} - K\right)^+\right]$$

$$\boxed{\mathbb{P}\{N(t_{j+1}) - N(t_j) = k\} = \frac{\lambda^k (t_{j+1}-t_j)^k}{k!} e^{-\lambda(t_{j+1}-t_j)}, \quad k = 0,1,\ldots. \qquad (11.2.7)}$$

$$= \sum_{j=0}^{\infty} e^{-r(T-t)}\left(xe^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^j - K\right)^+ \frac{\tilde{\lambda}^j (T-t)^j}{j!} e^{-\tilde{\lambda}(T-t)}$$

$$= \sum_{j=0}^{\infty} \left(xe^{-\tilde{\lambda}\sigma(T-t)}(\sigma+1)^j - Ke^{-r(T-t)}\right)^+ \frac{\tilde{\lambda}^j (T-t)^j}{j!} e^{-\tilde{\lambda}(T-t)}. \qquad (11.7.3)$$

From this formula, the risk-neutral price of the call $c(t,x)$ can be computed.

$$c(t,x) = \sum_{j=0}^{\infty} \left( x e^{-\tilde{\lambda}\sigma(T-t)} (\sigma+1)^j - K e^{-r(T-t)} \right)^+ \frac{\tilde{\lambda}^j (T-t)^j}{j!} e^{-\tilde{\lambda}(T-t)} \qquad (11.7.3)$$

The $j = 0$ term in (11.7.3) is

$$\left( x e^{-\tilde{\lambda}\sigma(T-t)} - K e^{-r(T-t)} \right)^+ e^{-\tilde{\lambda}(T-t)}.$$

When $t = T$, this term is $(x - K)^+$, and it is the only nonzero term in the sum in (11.7.3) when $t = T$. Therefore, the function $c$ satisfies the terminal condition

$$c(T,x) = (x - K)^+ \text{ for all } x \geq 0. \qquad (11.7.4)$$

# The partial differential equation that $c(t,x)$ must satisfy.

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S(t)e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N(T)-N(t)} - K\right)^+ \middle| \mathcal{F}(t)\right] = c(t, S(t))$$

$$e^{-rt}V(t) = \widetilde{\mathbb{E}}[e^{-rT}(S(T)-K)^+|\mathcal{F}(t)] = e^{-rt}c(t,S(t))$$

$e^{-rt}V(t) = \widetilde{\mathbb{E}}[e^{-rT}(S(T)-K)^+|\mathcal{F}(t)]$ is a martingale under $\widetilde{\mathbb{P}}$.

$\Rightarrow$ **$dt$ term of $d\left(e^{-rt}c(t,S(t))\right)$ should be "zero".**

# First, using Ito-Doeblin formula

**Theorem 11.5.4 (Two-dimensional Itô-Doeblin formula for processes with jumps).** *Let $X_1(t)$ and $X_2(t)$ be jump processes, and let $f(t, x_1, x_2)$ be a function whose first and second partial derivatives appearing in the following formula are defined and are continuous. Then*

$$
\begin{aligned}
& f(t, X_1(t), X_2(t)) \\
& = f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s))\, ds \\
& \quad + \int_0^t f_{x_1}(s, X_1(s), X_2(s))\, dX_1^c(s) + \int_0^t f_{x_2}(s, X_1(s), X_2(s))\, dX_2^c(s) \\
& \quad + \frac{1}{2}\int_0^t f_{x_1,x_1}(s, X_1(s), X_2(s))\, dX_1^c(s)\, dX_1^c(s) \\
& \quad + \int_0^t f_{x_1,x_2}(s, X_1(s), X_2(s))\, dX_1^c(s)\, dX_2^c(s) \\
& \quad + \frac{1}{2}\int_0^t f_{x_2,x_2}(s, X_1(s), X_2(s))\, dX_2^c(s)\, dX_2^c(s) \\
& \quad + \sum_{0<s\le t} \left[ f(s, X_1(s), X_2(s)) - f(s, X_1(s-), X_2(s-)) \right].
\end{aligned}
$$

$$dS(t) = (r - \tilde{\lambda}\sigma)S(t)\, dt + \sigma S(t-)\, dN(t),$$

$$dS^c(t) = (r - \tilde{\lambda}\sigma)S(t)\, dt.$$

$$\Delta S(t) = S(t) - S(t-) = \sigma S(t-),$$

$$S(t) = (\sigma + 1)S(t-).$$

**Theorem 11.5.4 (Two-dimensional Itô-Doeblin formula for processes with jumps).** *Let $X_1(t)$ and $X_2(t)$ be jump processes, and let $f(t, x_1, x_2)$ be a function whose first and second partial derivatives appearing in the following formula are defined and are continuous. Then*

$$
\begin{aligned}
& f(t, X_1(t), X_2(t)) \\
& = f(0, X_1(0), X_2(0)) + \int_0^t f_t(s, X_1(s), X_2(s))\, ds \\
& + \int_0^t f_{x_1}(s, X_1(s), X_2(s))\, dX_1^c(s) + \int_0^t f_{x_2}(s, X_1(s), X_2(s))\, dX_2^c(s) \\
& + \frac{1}{2}\int_0^t f_{x_1,x_1}(s, X_1(s), X_2(s))\, dX_1^c(s)\, dX_1^c(s) \\
& + \int_0^t f_{x_1,x_2}(s, X_1(s), X_2(s))\, dX_1^c(s)\, dX_2^c(s) \\
& + \frac{1}{2}\int_0^t f_{x_2,x_2}(s, X_1(s), X_2(s))\, dX_2^c(s)\, dX_2^c(s) \\
& + \sum_{0<s\le t} \left[ f(s, X_1(s), X_2(s)) - f(s, X_1(s-), X_2(s-)) \right].
\end{aligned}
$$

$$e^{-rt}c(t, S(t))$$

$$dS^c(t) = (r - \tilde{\lambda}\sigma)S(t)\, dt.$$

$$S(t) = (\sigma + 1)S(t-).$$

- $f(t, X_1(t)) \to e^{-rt}c(t, S(t))$
- $f(0, X_1(0)) \to c(0, S(0))$
- $f_t(s, X_1(s))\, ds \to \left[-re^{-ru}c(u, S(u)) + e^{-ru}c_t(u, S(u))\right] du$
- $f_{x_1}(s, X_1(s))\, dX_1^c(s) \to e^{-ru}c_x(u, S(u))\, dS^c(u)$
- $f_{x_1x_1}(s, X_1(s))\, dX_1^c(s)\, dX_1^c(s) \to 0$
- $\sum_{0\le s\le t}\left[f(s, X_1(s)) - f(s, X_1(s-))\right]$

  $\to \sum_{0\le u\le t}\left[e^{-ru}c(u, S(u)) - e^{-ru}c(u, S(u-))\right]$

  $= \sum_{0\le u\le t} e^{-ru}\left[c(u, S(u)) - c(u, S(u-))\right]$

$$
\begin{aligned}
e^{-rt}c(t,S(t)) \\
= c(0,S(0)) + \int_0^t e^{-ru}\big[-rc(u,S(u))\,du + c_t(u,S(u))\,du \\
+c_x(u,S(u))\,\underline{dS^c(u)}\big]
\end{aligned}
$$

$dS^c(u) = (r - \tilde{\lambda}\sigma)S(u)\,du$

$$
+\sum_{0<u\le t} e^{-ru}\big[c(u,\underline{S(u)}) - c(u,S(u-))\big]
$$

$S(u) = (\sigma+1)S(u-)$

$$
\begin{aligned}
= c(0,S(0)) + \int_0^t e^{-ru}\big[-rc(u,S(u)) + c_t(u,S(u)) \\
+\underline{(r-\tilde{\lambda}\sigma)S(u)}c_x(u,S(u))\big]\,\underline{du} \\
+\int_0^t e^{-ru}\big[c(u,\underline{(\sigma+1)S(u-)}) - c(u,S(u-))\big]\,dN(u)
\end{aligned}
$$

$$
\begin{aligned}
&e^{-rt}c(t,S(t)) \\
&\quad = c(0,S(0)) + \int_0^t e^{-ru}\big[-rc(u,S(u)) + c_t(u,S(u)) \\
&\qquad\qquad + (r-\tilde{\lambda}\sigma)S(u)c_x(u,S(u))\big]\,du \\
&\qquad + \int_0^t e^{-ru}\big[c(u,(\sigma+1)S(u-)) - c(u,S(u-))\big]\,\underline{dN(u)} \\
&\quad = c(0,S(0)) + \int_0^t e^{-ru}\big[-rc(u,S(u)) + c_t(u,S(u)) \\
&\qquad\qquad + (r-\tilde{\lambda}\sigma)S(u)c_x(u,S(u))\big]\,du \\
&\qquad + \int_0^t e^{-ru}\big[c(u,(\sigma+1)S(u-)) - c(u,S(u-))\big]\,\underline{\tilde{\lambda}\,du} \\
&\qquad + \int_0^t e^{-ru}\big[c(u,(\sigma+1)S(u-)) - c(u,S(u-))\big]\,\underline{d\widetilde{M}(u).}
\end{aligned}
$$

$\widetilde{M}(u) = N(u) - \tilde{\lambda} u$

$d\widetilde{M}(u) = dN(u) - \tilde{\lambda} du$

$dN(u) = \tilde{\lambda} du + d\widetilde{M}(u)$

However, the integral

$$\int_0^t e^{-ru}\big[c(u,(\sigma+1)\underline{S(u-)}) - c(u,\underline{S(u-)})\big]\tilde{\lambda}\,du$$

is the same as the integral

$$\int_0^t e^{-ru}\big[c(u,(\sigma+1)\underline{S(u)}) - c(u,\underline{S(u)})\big]\tilde{\lambda}\,du.$$

We have shown that

$$\begin{aligned}
&e^{-rt}c(t,S(t))\\
&= c(0,S(0))\\
&\quad + \int_0^t e^{-ru}\Big[-rc(u,S(u)) + c_t(u,S(u)) + (r-\tilde{\lambda}\sigma)S(u)c_x(u,S(u))\\
&\qquad\qquad + \tilde{\lambda}\big(c(u,(\sigma+1)S(u)) - c(u,S(u))\big)\Big]\,du\\
&\quad + \int_0^t e^{-ru}\big[c(u,(\sigma+1)S(u-)) - c(u,S(u-))\big]\,d\widetilde{M}(u). \qquad (11.7.6)
\end{aligned}$$

$$
\begin{aligned}
&\underline{e^{-rt}c(t,S(t))} \quad \text{Martingale}\\
&= c(0,S(0))\\
&\quad + \int_0^t e^{-ru}\Big[-rc(u,S(u)) + c_t(u,S(u)) + (r-\tilde{\lambda}\sigma)S(u)c_x(u,S(u))\\
&\qquad + \tilde{\lambda}\big(c(u,(\sigma+1)S(u)) - c(u,S(u))\big)\Big]\,du\\
&\quad + \underbrace{\int_0^t e^{-ru}\big[c(u,(\sigma+1)S(u-)) - c(u,S(u-))\big]\,d\widetilde{M}(u).}_{\text{Martingale}} \qquad (11.7.6)
\end{aligned}
$$

**Theorem 11.4.5.** *Assume that the jump process $X(s)$ of (11.4.1)–(11.4.3) is a martingale, the integrand $\Phi(s)$ is* left-continuous *and adapted, and*

$$\mathbb{E}\int_0^t \Gamma^2(s)\Phi^2(s)\,ds < \infty \text{ for all } t \geq 0.$$

*Then the stochastic integral $\int_0^t \Phi(s)\,dX(s)$ is also a martingale.*

$$c(0,S(0)) + \int_0^t e^{-ru}\Big[-rc(u,S(u)) + c_t(u,S(u)) + (r-\tilde{\lambda}\sigma)S(u)c_x(u,S(u))$$
$$+\tilde{\lambda}\big(c(u,(\sigma+1)S(u)) - c(u,S(u))\big)\Big]\,du$$

and see that it is the difference of two martingales and hence is itself a martingale. This can only happen if the integrand is zero:

$$-rc(t,S(t)) + c_t(t,S(t)) + (r-\tilde{\lambda}\sigma)S(t)c_x(t,S(t))$$
$$+\tilde{\lambda}\big(c(t,(\sigma+1)S(t)) - c(t,S(t))\big) = 0. \quad (11.7.7)$$

$$
\begin{aligned}
&e^{-rt}c(t,S(t))\\
&= c(0,S(0))\\
&\quad + \int_0^t e^{-ru}\Big[-rc(u,S(u)) + c_t(u,S(u)) + (r-\tilde{\lambda}\sigma)S(u)c_x(u,S(u))\\
&\qquad\qquad + \tilde{\lambda}\big(c(u,(\sigma+1)S(u)) - c(u,S(u))\big)\Big]\,du\\
&\quad + \int_0^t e^{-ru}\big[c(u,(\sigma+1)S(u-)) - c(u,S(u-))\big]\,d\widetilde{M}(u). \qquad (11.7.6)
\end{aligned}
$$

$$
\begin{aligned}
&d\big(e^{-rt}c(t,S(t))\big)\\
&= e^{-rt}\Big[-rc(t,S(t)) + c_t(t,S(t)) + (r-\tilde{\lambda}\sigma)S(t)c_x(t,S(t))\\
&\qquad + \tilde{\lambda}\big(c(t,(\sigma+1)S(t)) - c(t,S(t))\big)\Big]\,dt\\
&\qquad + e^{-rt}\big[c(t,(\sigma+1)S(t-)) - c(t,S(t-))\big]\,d\widetilde{M}(t)
\end{aligned}
$$

setting the $dt$ term equal to zero.

We conclude by replacing the stock price process $S(t)$ in (11.7.7) by a dummy variable $x$. This gives the equation

$$-rc(t,x)+c_t(t,x)+(r-\tilde{\lambda}\sigma)xc_x(t,x)+\tilde{\lambda}\big(c(t,(\sigma+1)x)-c(t,x)\big)=0, \quad (11.7.9)$$

which must hold for $0 \le t < T$ and $x \ge 0$. This is sometimes called a *differential-difference* equation because it involves $c$ at two different values of the stock price, namely $x$ and $(\sigma+1)x$. The function $c(t,x)$ defined by (11.7.3) satisfies this equation because, by its construction, $e^{-rt}c(t,S(t))$ is a martingale under $\widetilde{\mathbb{P}}$.

Returning to (11.7.6) and using equation (11.7.9), we see that for $0 \leq t \leq T$,

$$e^{-rt}c(t, S(t))$$
$$= c(0, S(0)) + \int_0^t e^{-ru}\left[c(u, (\sigma+1)S(u-)) - c(u, S(u-))\right] d\widetilde{M}(u). \quad (11.7.10)$$

In particular,

$$e^{-rT}(S(T) - K)^+$$
$$= e^{-rT}c(T, S(T))$$
$$= c(0, S(0)) + \int_0^T e^{-ru}\left[c(u, (\sigma+1)S(u-)) - c(u, S(u-))\right] d\widetilde{M}(u). \quad (11.7.11)$$